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MICHAELE CIPOLLA

SPECIMEN

DE

CALCULO ARITHMETICO-INTEGRALE

PARTE I

Operationes fundamentale

Typographia Cooperativa, cum typos de « Revista de Mathematica ».

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IN MEMORIA

DE

E. CESÀRO

REVISTA DE MATHEMATICA
TOMO IX A. 1908.

MICHAELE CIPOLLA

SPECIMEN DE CALCULO ARITHMETICO-INTEGRALE

Parte 1. Operationes fundamentale

PRÆFATIO

> Prendo un sasso, tra mille, a quando a quando:
> lo netto, arroto, taglio, lustro, affino:
> chi mi sia, non importa: ecco un rubino;
> vedi un topazio, prendi un ametista.
>
> G. PASCOLI.

Studio de functiones numerico es iam disciplina de magno momento in Analysi mathematico moderno.

Theoria de valore medio et de expressione asymptotico, creato a mente geniali de LEJEUNE-DIRICHLET, es materia de pulchro et profundo investigationes per TCHEBYCHEF, RIEMANN, LIOUVILLE, MERTENS, GEGENBAUER, von MANGOLDT, DE LA VALLÉE-POUSSIN, LANDAU, CESÀRO, TORELLI et alios, et hodie plure juvenili ingenio et forte et vegeto versa in tali studio cum ardore et animo.

Omni investigatione super functiones numerico et in particulare modo super lege de suo variatione, trahe origine ex identitate que subsiste inter functiones numerico. Nemo in vero dubita que studio de proprietates *ad finito*, ut nos ita dic, debe præcede studio de proprietates *ad infinito*, et de hoc studio es vero et securo fundamento.

Sane eveni ut, per ignorantia de relationes inter functiones numerico, plure proprietate de que omnes præsentiba veritate remane-vi per longo tempore sine ullo demonstratione, contra postea inventio de illo relationes duce-vi sæpe non solo ad demonstratione desiderato sed et ad veritate inexspectato. Itaque cum optimo jure Cesàro in uno de suo primo scripto arithmetico dic: *pour nous l'avenir de l'Arithmétique est une habile combinaison de fonctions habilement choisies.*

Ergo oporte antea ut combina functiones numerico in modo opportuno, stude relationes que resulta ex tali combinatione et postea relinque ad Analysi infinitesimale to oppugna et supera omni difficultate circa lege de variatione et inveni proprietate magis recondito: tali es incepto de plure insigni Mathematico et in particulare modo de Bougajeff et Cesàro.

Sed investigatione super identitates arithmetico es disperso in plure diurnale sine ullo systema; demonstrationes et es ducto cum vario methodo sæpe contorto et difficili. De hoc bene senti Cesàro, ad que pertine primo conatu ut introduc opportuno symbolo de operatione.

Nunc nos funda theoria de identitates inter functiones numerico super solo notione de *producto integrale*. Analogia de ipso cum producto arithmetico et introductione de opportuno symbolo conduce ad calculo operativo multo simplice ex que plure theorema novo et proprietates noto inveni demonstratione multo elegante, antea insperato.

Tali es nostro calculo arithmetico integrale de que nos publica nunc hoc parvo specimen que tracta solo de operationes fundamentale.

Si favore et benevolentia de illos que dilige et excole theoria de numeros adde animo ad nos in conficiendo opere, nos spera ede postea parte que tracta de Applicationes, que es multo et interessante.

Hic nos, per opportuno extensione de notione de producto integrale, solo monstra parte que habe nostro novo functiones numerico in evolutione de functione in serie de Dirichlet.

Hoc parvo scripto es dedicato ad memoria de E. Cesàro, que pro Theoria de numeros impende-vi cum magno efficacitate tanto parte de suo opera geniali et fecundo et que sæpe accende-vi animo nostro cum verbo et exemplo.

In Panormo, die 31 julio 1907.

Michaele Cipolla.

§ 1. **Producto integrale**

✻ 1. $f,g,h\varepsilon\, \mathrm{q}'\mathrm{FN}_1 \,.\, x\varepsilon \mathrm{N}_1 \,.\supset.$

·0 $(f\mathrm{x}g)x = \Sigma[fx_1 \times gx_2 | (x_1,x_2),\ (x_1,x_2)\ni(x_1,x_2\varepsilon\mathrm{N}_1 \,.\, x_1x_2 = x)]$
$= \Sigma[fa \times g(x/a) | a,\ \mathrm{N}_1 \frown x/\mathrm{N}_1]$ Df x

·01 $f\mathrm{x}g \ = [(f\mathrm{x}g)x \mid x,\ \mathrm{N}_1]$

$\mathrm{q}'\mathrm{FN}_1$ es appellato « functione numerico » (*Zahlentheoretische Function*).

Lege $(f\mathrm{x}g)x$ « producto integrale de f per g, relativo ad x ». Illo es summa de omni producto $fx_1 \times fx_2$, extenso ad omni solutione integro et positivo de æquatione $x_1x_2 = x$.

Particulare producto integrale es considerato iam a Dedekind (J. f. Math., t.54, a.1857), a Liouville (J. de Math., s.2, a.1857) et ab alios. Postea Bougajeff publica plure scripto super productos integrale (J. phil., t.5, a.1871; Collect. math. de Mosca, in russo, t.14, a.1888, p.1-45; t.14, a.1889, p.169-197; t.18, a.1895, p.720-758; t.18, a.1896, p1-54; t.20, a.1898-99, p.549-594; t.21, a. 1900, p.335-350; t.21, a. 1900, p.499-536). Cesàro iam initia studio accurato de producto integrale et adopta signo $*$, sed scribe in modo minus simplice $\int fx * gx$, et appella illo « integrale composito » (v. Bruxelles, a.1883, p.352 = Mém. de la Soc. Roy. de Liège, s.2, t.11; Giorn. di Battaglini, t.23, a.1885, p.168-174, p.175-181, et alio scriptos. Signo $*$ es in *Nuova contribuzione ai principî fondamentali dell'Aritmetica assintotica*, Atti della R. Acc, d. s. f. e mat. di Napoli, s.2, t.6, a.1894 p.23).

Nos adopta denominatione de producto integrale et appella operatione « multiplicatione integrale », quare signo x habe proprietates analogo de signo $\times$ de multiplicatione arithmetico. In vero ex Df x seque

·1 $f\mathrm{x}g \ \varepsilon\ \mathrm{q}'\mathrm{FN}_1$

·2 $f\mathrm{x}(g+h) = f\mathrm{x}g + f\mathrm{x}h$ Distrib(x, +)

·3 $(f+g)\mathrm{x}h = f\mathrm{x}h + g\mathrm{x}h$ Distrib(+, x)

·4 $f\mathrm{x}g = g\mathrm{x}f$ Comm x

$\cdot 5 \quad (f\mathrm{x}g)\mathrm{x}h = f\mathrm{x}(g\mathrm{x}h)$ Assoc x

$\cdot 51 \quad f\mathrm{x}g\mathrm{x}h = (f\mathrm{x}g)\mathrm{x}h$ Df

P1·5 resulta et ex P sequente

$\cdot 6 \quad (f\mathrm{x}g\mathrm{x}h)x =$

$\Sigma[fx_1 \times gx_2 \times hx_3 | (x_1,x_2,x_3), (\mathrm{N}_1 ; \mathrm{N}_1 ; \mathrm{N}_1 ;) \cap (x_1,x_2,x_3) \ni (x_1 x_2 x_3 = x)]$ Dfp

que nos pote sume ut Df de producto integrale de tres et, in modo analogo, de plure functiones numerico.

$\cdot 7 \quad f1 -= 0 \,.\, f\mathrm{x}g = f\mathrm{x}h \,.\supset.\, g = h$

Ce P seque in modo facile per lege de inductione.

§ 2. **Functiones numerico imprimitivo, composito, logarithmico**

✻ $2\cdot 0 \quad (\mathrm{q}'\mathrm{FN}_1)\mathrm{impr} =$
$(\mathrm{q}'\mathrm{FN}_1) \cap f \ni [x,y \,\varepsilon\, \mathrm{N}_1 \,.\, \mathrm{D}(x,y) = 1 \,.\supset_{x,y}.\, f(x \times y) = fx \times fy]$ Df

$\cdot 01 \quad (\mathrm{q}'\mathrm{FN}_1)\mathrm{comp} =$
$(\mathrm{q}'\mathrm{FN}_1) \cap f \ni [x,y \,\varepsilon\, \mathrm{N}_1 \,.\supset_{x,y}.\, f(x \times y) = fx \times fy]$ Df

Lege $(\mathrm{q}'\mathrm{FN}_1)\mathrm{impr}$ « functione numerico imprimivo » et $(\mathrm{q}'\mathrm{FN}_1)\mathrm{comp}$ « functione numerico composito ».

Seque ex Df:

$\cdot 1 \quad (\mathrm{q}'\mathrm{FN}_1)\mathrm{comp} \supset (\mathrm{q}'\mathrm{FN}_1)\mathrm{impr}$

$\cdot 2 \quad f \varepsilon (\mathrm{q}'\mathrm{FN}_1)\mathrm{impr} \,.\supset.\, f1 \,\varepsilon\, \iota 0 \cup \iota 1$

P·2 in phrasi: omni functione imprimitivo pro valore 1 de variabile sume aut valore 0 aut valore 1.

Dem. $\quad \mathrm{Hp} \,.\, f1 = f(1 \times 1) \,.\, \mathrm{Def}\cdot 0 \,.\supset.\, f1 = (f1)^2 \,.\supset.\, \mathrm{P}.$

Propositiones sequente completa P·2.

$\cdot 21 \quad \mathrm{Hp}\cdot 2 \,.\, f1 = 0 \,.\supset:\, x \varepsilon \mathrm{N}_1 \,.\supset_x.\, fx = 0$

$\cdot 22 \quad \mathrm{Hp}\cdot 2 \,.\, \exists\, \mathrm{N}_1 \cap x \ni (fx -= 0) \,.\supset.\, f1 = 1$

P·21. Dm. $\quad \mathrm{Hp}\cdot 21.\, x \varepsilon \mathrm{N}_1 \,.\supset_x.\, fx = f(1 \times x) = f1 \times fx = 0.$
P·22. Dm. $\quad \mathrm{Hp}\cdot 22.\, x \varepsilon \mathrm{N}_1 .\, fx -= 0 \,.\supset.\, fx \times f1 = fx -= 0 \,.\supset.\, f1 -= 0.$

Valores de functione numerico composito es noto si es noto valores de ipso pro valore 1 et pro omni numero primo; et valore que sume functione numerico imprimitivo es noto si es noto valores de ipso pro valore 1 et pro omni potestate de numero primo.

In casu particulare subsiste Distrib(×,x), ut monstra P sequente

$\cdot 3 \quad f \varepsilon (\mathrm{q}'\mathrm{FN}_1)\mathrm{comp} \,.\, g,h \varepsilon\, \mathrm{q}'\mathrm{FN}_1 \,.\supset.\, f \times (g\mathrm{x}h) = (f \times g)\mathrm{x}(f \times h)$

Dm. de ce P seque subito ex Df x et P·01.

Exemplo de functiones composito:

$\cdot 4 \quad m \varepsilon \mathrm{q} \,.\supset.\, [x^m | x, \mathrm{N}_1] \,\varepsilon\, (\mathrm{q}'\mathrm{FN}_1)\mathrm{comp}$

$\cdot 5 \quad [\sin(\pi x/2) | x, \mathrm{N}_1] \,\varepsilon\, (\mathrm{q}'\mathrm{FN}_1)\mathrm{comp}$

Nos inveni mox plure functione imprimitivo non composito.

✻ $3\cdot 0 \quad (\mathrm{q}'\mathrm{FN}_1)\log =$
$(\mathrm{q}'\mathrm{FN}_1) \cap f \ni [x,y \,\varepsilon\, \mathrm{N}_1 \,.\, \mathrm{D}(x,y) = 1 \,.\supset_{x,y}.\, f(x \times y) = fx + fy]$ Df

Lege $(\mathrm{q}'\mathrm{FN}_1)\log$ « functione numerico logarithmico »; denominatione adoptato a Bougajeff (Collect. Math. de Mosca, in russo, t.13, a.1887, p.757-777). Ex ce Df. seque subito

$\cdot 1 \quad f \varepsilon (\mathrm{q}'\mathrm{FN}_1)\log \,.\supset.\, f1 = 0$

$\cdot 2 \quad f \varepsilon ((\mathrm{q}' - \iota 0)\mathrm{FN}_1)\mathrm{impr} \,.\supset.\, \log f \,\varepsilon\, (\mathrm{q}'\mathrm{FN}_1)\log$

$\cdot 3 \quad m \varepsilon \mathrm{N}_1 \,.\supset.\, \varkappa m = \mathrm{num}(\mathrm{Np} \cap m/\mathrm{N}_1)$ Df $\varkappa$

$\cdot 4 \quad$ —— $\,.\supset.\, \tau m = \Sigma[\mathrm{mp}(x,m) | x, \mathrm{Np}]$ Df τ

$\varkappa m$ = numero de divisore primo de m, τm = summa de maximo exponente de potestate de numero primo que divide m. Es duo functione numerico frequente in nostro studio.

$\cdot 5 \quad \varkappa, \tau \,\varepsilon\, (\mathrm{q}'\mathrm{FN}_1)\log$

§ 3. **Functione α vel unitate integrale.**

Notabile es functione numerico α definito in modo sequente

✻ $4\cdot 0 \quad x \varepsilon\, \mathrm{N}_1 + 1 \,.\supset.\, \alpha 1 = 1 \,.\, \alpha x = 0$ Df α

Seque subito

·1 $a \; \varepsilon \; (\mathrm{q}'\mathrm{FN}_1)\mathrm{comp}$

·2 $f\varepsilon \; \mathrm{q}'\mathrm{FN}_1 \; .\supset. \; f\mathrm{x}a = f$

Ergo functione a habe in multiplicatione integrale idem parte que numero 1 in multiplicatione arithmetico; per tali proprietate nos appella a « unitate integrale ». CESÀRO voca illo « functione fundamentale » et indica illo per symbolo ε.

Si nunc nos pone Df sequente

·3 $a,b \; \varepsilon \; \mathrm{Cls}'(\mathrm{q}'\mathrm{FN}_1) \; .\supset.$
$a\mathrm{x}b = f\ni[\exists(h,k)\ni(h\varepsilon a \,.\, k\varepsilon b \,.\, f = h\mathrm{x}k)]$ Df

tunc nos deduce theorema notabile

·4 $(\mathrm{q}'\mathrm{FN}_1)\mathrm{impr} \; \mathrm{x} \; (\mathrm{q}'\mathrm{FN}_1)\mathrm{impr} = (\mathrm{q}'\mathrm{FN}_1)\mathrm{impr}$

in phrasi: *producto integrale de functione numerico imprimitivo es functione numerico imprimitivo.*

Dem.

$m,n\varepsilon \mathrm{N}_1 \,.\, \mathrm{D}(m,n) = 1 \,.\, f,g \; \varepsilon \; (\mathrm{q}'\mathrm{FN}_1)\mathrm{impr} \; .\supset.$
$(f\mathrm{x}g)m \times (f\mathrm{x}g)n = \Sigma[fr\times g(m/r)|r, \mathrm{N}_1\frown m/\mathrm{N}_1] \times \Sigma[fs\times g(n/s)|s, \mathrm{N}_1\frown n/\mathrm{N}_1]$
$= \Sigma[\Sigma\{fr\times fs\times g(m/r)\times g(n/s)|r, \mathrm{N}_1\frown m/\mathrm{N}_1\}|s, \mathrm{N}_1\frown n/\mathrm{N}_1]$ (1)
$m,n\varepsilon \mathrm{N}_1 \,.\, \mathrm{D}(m,n) = 1 \,.\, r\varepsilon \; \mathrm{N}_1\frown m/\mathrm{N}_1 \,.\, s\varepsilon \; \mathrm{N}_1\frown n/\mathrm{N}_1 \; .\supset. \; \mathrm{D}(r,s) = 1$ (2)
$(1) \,.\, (2) \; .\supset. \; (f\mathrm{x}g)m \times (f\mathrm{x}g)n = \Sigma[\Sigma\{f(rs)\times g(mn/(rs))|r, \mathrm{N}_1\frown m/\mathrm{N}_1\}|s, \mathrm{N}_1\frown n/\mathrm{N}_1]$ (3)
$m,n\varepsilon \mathrm{N}_1 \,.\, \mathrm{D}(m,n) = 1 \; .\supset. \; \mathrm{N}_1\frown mn/\mathrm{N}_1 = (\mathrm{N}_1\frown m/\mathrm{N}_1) \times (\mathrm{N}_1\frown n/\mathrm{N}_1)$ (4)
$(3) \,.\, (4) \; .\supset. \; (f\mathrm{x}g)m \times (f\mathrm{x}g)n = \Sigma[fx\times g(mn/x)|x, \mathrm{N}_1\frown mn/\mathrm{N}_1]$ (5)
$(5) \,.\, \mathrm{P}{\cdot}2 \; .\supset. \; \mathrm{P}$

§ 4 Functione v. Integrale numerico. Functione ν et σ.

Alio functione notabile es illo que habe semper valore 1 et que nos denota cum symbolo v:

✻ 5·0 $x\varepsilon \mathrm{N}_1 \; .\supset. \; vx = 1$ Df v

·1 $v\varepsilon \; (\mathrm{q}'\mathrm{FN}_1)\mathrm{comp}$

·2 $f\varepsilon \; \mathrm{q}'\mathrm{FN}_1 \; .\supset. \; f\mathrm{x}v = f$

Producto integrale de functione numerico f per functione v

es appellato a BOUGAJEFF et CESÀRO « integrale numerico de f », et CESÀRO, que extende notatione de EULERO (in casu particulare $\int x = \Sigma(\mathrm{N}_1\frown x/\mathrm{N}_1)$., EULERO, Opusc. v. arg. 2, a.1750, pagina 23 = Comm. Art.1 p.102), denota illo cum symbolo $\int f$. Nos adopta tali denominatione et notatione.

·3 $f\varepsilon \; \mathrm{q}'\mathrm{FN}_1 \; .\supset. \; \int f = f\mathrm{x}v$ Df

·34 ——— . $x\varepsilon \mathrm{N}_1 \; .\supset. \; \int fx = (\int f)x$ Df

·4 ——— . — $.\supset. \; \int fx = \Sigma(f, \mathrm{N}_1\frown x/\mathrm{N}_1)$

Ex P4·4 seque subito

·5 $f\varepsilon \; (\mathrm{q}'\mathrm{FN}_1)\mathrm{impr} \; .\supset. \; \int f \; \varepsilon \; (\mathrm{q}'\mathrm{FN}_1)\mathrm{impr}$

·51 ———— . $x\varepsilon \; \mathrm{N}_1 \; .\supset.$

$\int fx = \Pi[\int f(r\upharpoonleft\mathrm{mp}(r,x))|r, \mathrm{Np}\frown x/\mathrm{N}_1]$

Si nunc nos pone

✻ 6·0 $\nu = \int v$ Df

·01 $m\varepsilon\mathrm{q} \,.\, x\varepsilon \mathrm{N}_1 \; .\supset. \; \sigma_m = \int(x^m|x)$ Df

·02 $\sigma = \sigma_1$ Df

et nos observa que per P 5·1, 2·4, 5·4

·1 $\nu,\sigma_m\varepsilon \; (\mathrm{q}'\mathrm{FN}_1)\mathrm{impr}$

nos subito deduce ex P·41 propositiones noto sequente

·2 $x\varepsilon \mathrm{N}_1 \; .\supset. \; \nu x = \mathrm{num}(\mathrm{N}_1\frown x/\mathrm{N}_1) = \Pi\{[\mathrm{mp}(r,x)+1]|r, \mathrm{Np}\frown x/\mathrm{N}_1\}$

·3 —— $.\supset. \; \sigma x = \Sigma(\mathrm{N}_1\frown x/\mathrm{N}_1)$
$= \Pi\{\Sigma[a^r|r, 0\cdots\mathrm{mp}(a,x)]|a, \mathrm{Np}\frown x/\mathrm{N}_1\}$
$= \Pi[\{(a\upharpoonleft[\mathrm{mp}(a,x)+1]-1)/(a-1)\}|a, \mathrm{Np}\frown x/\mathrm{N}_1\}$

[V. F1905, III, §16, P5·2, 5·3]

·4 $a,x\varepsilon \mathrm{N}_1 \; .\supset. \; \int(a\upharpoonleft\varkappa x|x)x = \nu(x^a)$

·5 $x\varepsilon \mathrm{N}_1 \; .\supset. \; \int\nu x = \nu x\times\Sigma[/(\mathrm{mp}(r,x)+1)|r, \mathrm{Np}\frown x/\mathrm{N}_1]$

Pro calculo de producto integrale de uno functione imprimitivo per functione logarithmico es utile P sequente

·6 $f\varepsilon \; (\mathrm{q}'\mathrm{FN}_1)\mathrm{imp} \,.\, l\varepsilon \; (\mathrm{q}'\mathrm{FN}_1)\log \,.\, x\varepsilon \; \mathrm{N}_1+1 \; .\supset.$
$(f\mathrm{x}l) = \Sigma\{[(f\mathrm{x}l)(a\upharpoonleft\mathrm{mp}(a,x))] \times \int f(x/[a\upharpoonleft\mathrm{mp}(x,a)])|a, \mathrm{Np}\frown x/\mathrm{N}_1\}$

ex que nos deduce

$\cdot 7 \quad l \varepsilon\, (\mathrm{q}'\mathrm{FN}_1)\mathrm{log} \,.\, x \varepsilon \mathrm{N}_1+1 \,.\supset.$
$\int lx = \nu x \times \Sigma\{[\int l(r \upharpoonright \mathrm{mp}(r,x))]/(\mathrm{mp}(r,x)+1) | r,\ \mathrm{Np} \cap x/\mathrm{N}_1\}$
$[\ (\nu \mid f)\ \mathrm{P}\cdot 6 \,.\supset.\ \mathrm{P}\ ]$

✻ $7. \quad f \varepsilon\, \mathrm{q}'\mathrm{FN}_1 \,.\, m \varepsilon \mathrm{N}_1 \,.\supset.$

$\cdot 0 \quad \int^1 f = \int f \,.\, \int^{m+1} f = \int^m \int f$ Df

Lege $\int^m f$ « integrale numerico de f de ordine m ».

Pro calculo de valore de integrale multiplo, in particulare de functione imprimitivo, es utile P sequente, que pote demonstrare per lege de inductione:

$\cdot 1 \quad n \varepsilon \mathrm{N}_1 \,.\, a \varepsilon \mathrm{Np} \,.\supset.$
$\int^m f(a^n) = \Sigma[\mathrm{C}(m+r-1,\ r) \times f(a^{n-r}) | r,\ 0 \cdots n]$

§ 5. **Potestate integrale vel cointegrale**

✻ $8. \quad r,s \varepsilon \mathrm{N}_0 \,.\, f,g \varepsilon\, \mathrm{q}'\mathrm{FN}_1 \,.\supset.$

$\cdot 0 \quad f\mathrm{x}^0 = \alpha \,.\, f\mathrm{x}^{r+1} = (f\mathrm{x}^r)\mathrm{x}f$ Df

Lege $f\mathrm{x}^r$ « potestate integrale » vel « cointegrale de f de gradu r ».

In facili modo pote demonstrare P sequentes

$\cdot 1 \quad (f\mathrm{x}g)\mathrm{x}^r = (f\mathrm{x}^r)\mathrm{x}(g\mathrm{x}^r)$

$\cdot 2 \quad (f\mathrm{x}^r)\mathrm{x}(f\mathrm{x}^s) = f\mathrm{x}^{r+s}$

$\cdot 3 \quad (f\mathrm{x}^r)\mathrm{x}^s = f\mathrm{x}^{rs}$

$\cdot 4 \quad (f+g)\mathrm{x}^r = \Sigma\{\mathrm{C}(r,n) \times [(f\mathrm{x}^{r-n})\mathrm{x}(g\mathrm{x}^n)] | n, \mathrm{N}_0\}$

Ergo potestate integrale habe proprietates analogo de potestate de numero.

$\cdot 5 \quad f \varepsilon\, (\mathrm{q}'\mathrm{FN}_1)\mathrm{impr} \,.\supset.\ f\mathrm{x}^r \varepsilon\, (\mathrm{q}'\mathrm{FN}_1)\mathrm{impr}$

Per ce P calculo de cointegrale de functione imprimitivo es reducto ad calculo de valore de cointegrale de variabile relativo ad potestate de numero primo.

In tali casu calculo pote fi per algorithmo isobaryco (v. CESÀRO, Analisi Algebrica, p.361), ut in P sequente

$\cdot 6 \quad p \varepsilon \mathrm{Np} \,.\, r,n \varepsilon \mathrm{N}_1 \,.\supset.$
$f\mathrm{x}^r(p^n) = \Sigma[\{(f1)^{r-m} \times \Sigma[\Pi(f(p^{xs}) | s, 1 \cdots m) | x,\ (\mathrm{N}_1\mathrm{F}1 \cdots m) \cap x \ni$
$(\Sigma x = n)]\} | m,\ 1 \cdots r]$

ubi si f es functione imprimitivo et non habe semper valore nullo, es $f1 = 1$ (P2·22). In particulare

$\cdot 7 \quad \mathrm{Hp}\cdot 6 \,.\supset.\ \nu\mathrm{x}^r(p^n) = \mathrm{num}\,[(\mathrm{N}_0\mathrm{F}1 \cdots r) \cap x \ni (\Sigma x = n)] =$
$= \mathrm{C}(r+n-1, n)$

[v. Form. 1905, III, §15, P17·1, pag. 127]

Si f es functione numerico composito, P·6 fi
$f\mathrm{x}^r(p^n) = f(p^n) \times \nu\mathrm{x}^r(p^n) = f(p^n) \times \mathrm{C}(r+n-1, n)$,
ergo in generali

$\cdot 8 \quad f \varepsilon\, (\mathrm{q}'\mathrm{FN}_1)\mathrm{comp} \,.\supset.\ f\mathrm{x}^r = f \times \nu\mathrm{x}^r$

$\cdot 81 \quad \mathrm{Hp}\cdot 8 \,.\supset.\ f\mathrm{x}^2 = f \times \nu$

Nos nunc demonstra P sequente, que exprime notabile relatione intra cointegrales de uno functione numerico:

✻ $9. \quad m,n \varepsilon \mathrm{N}_1 \,.\, n > \tau m \,.\, f \varepsilon\, \mathrm{q}'\mathrm{FN}_1 \,.\supset.$

$\cdot 0 \quad \Sigma[(-f1)^{n-r} \times \mathrm{C}(n,r) \times f\mathrm{x}^r m | r, \mathrm{N}_0] = 0$

Dem.

$\mathrm{Hp} \,.\, \mathrm{P}8\cdot 4 \,.\supset.\ (f-(f1)\times\alpha)\mathrm{x}^n m = \Sigma[(-f1)^{n-r} \times \mathrm{C}(n,r) \times f\mathrm{x}^r m | r, \mathrm{N}_0]$ (1)
$x \varepsilon \mathrm{N}_1+1 \,.\supset.\ (f-(f1)\times\alpha)1 = 0 \,.\, (f-(f1)\times\alpha)m = fm$ (2)
$\mathrm{Hp} \,.\, (2) \,.\supset.\ (f-(f1)\times)\alpha\mathrm{x}^n m =$
$\Sigma[\Pi(fx_r | r, 1 \cdots n) | x,\ ((\mathrm{N}_1+1)\mathrm{F}1 \cdots n) \cap x \varepsilon (\Pi x = m)]$ (3)
$\mathrm{Hp} \,.\, n > \tau m \,.\supset\ -\exists[(\mathrm{N}_1+1)\mathrm{F}1 \cdots n] \cap x \ni (\Pi x = m)$ (4)
$(3) \,.\, (4) \,.\supset.\ (f-(f1)\times\alpha)\mathrm{x}^n m = 0$ (5)
$(1) \,.\, (5) \,.\supset.\ \mathrm{P}$

Si nos effice producto integrale de functione g per duo membro de æqualitate ·0, nos obtine P sequente

$\cdot 1 \quad g \varepsilon \mathrm{q}'\mathrm{FN}_1 \,.\supset.$
$(-f1)^n \times gm = -\Sigma\{(-f1)^{n-r} \times \mathrm{C}(n,r) \times [(f\mathrm{x}^r)(\mathrm{x}g)m] | r, \mathrm{N}_1\}$

Si in P præcedente nos muta f in v, seque

$\cdot 2 \quad \mathrm{Hp}\cdot 1 \,.\supset.\ gm = \Sigma[(-1)^{r+1} \times \mathrm{C}(n,r) \times \int^r gm | r, \mathrm{N}_1]$

Calculo de valores de cointegrale de gradu r de functione numerico f pote es deducto ex valores de cointegrale de functione $f-(f1)\times a$. In vero si nos sume cointegrale de gradu r de duo membro de identitate $f=(f-(f1)\times a)+(f1)\times a$, nos obtine P sequente

$\cdot 3 \quad r\varepsilon N_1 \,.\supset.\, f\mathrm{x}^r = \Sigma[C(r,s)\times(f1)^{r-s}\times(f-(f1)\times a)\mathrm{x}^s \,|s, N_0]$

P. præcedente es utile, quare calculo de cointegrale de functione $f-(f1)\times a$ es magis simplice que calculo de cointegrales de f.

In vero, si nos observa que functione $f-(f1)\times a$ es nullo pro valore 1 de variabile, seque ex Df de cointegrale que $(f-(f1)\times a)\mathrm{x}^r\, m$ es æquale ad summa de omni producto de forma $(fx_1)\times(fx_2)\times\ldots(fx_r)$, correspondente ad omni solutione in numero integro et positivo et *majore de unitate* de æquatione $x_1x_2\ldots x_r=m$. In particulare, si es $f=v$, cointegrale de gradu r de $(v-a)m$ da numero de solutione in numero integro et majore de unitate de æquatione $x_1x_2\ldots x_r=m$, et per evolutione de $(v-a)\mathrm{x}^r$ et per P8·7, nos obtine:

$\cdot 4 \quad r,m\varepsilon N_1 \,.\, \mathrm{num}[(N_1+1)\, F\, 1\cdots r \cap x\ni(\Pi x=m)] =$
$\Sigma\{(-1)^s\times C(r,s)\times \Pi[C(r-s+\mathrm{mp}(x,m), r-s)\,|x, \mathrm{Np}\cap m/N_1]\,|s, N_0\}$

§ 6. **Functiones coniugato.**
Potestate integrale de gradu negativo.

Si functione numerico f non sume valore nullo pro valore 1 de variabile, tunc existe uno et solo functione numerico, que nos indica cum symbolo $f\mathrm{x}^{-1}$ et appella « functione coniugato de f », tali que producto integrale de ipso per f es unitate integrale, id es: $f\mathrm{x}(f\mathrm{x}^{-1})=a$.

In vero ex ce Df seque $f\mathrm{x}^{-1}1=/f1$ et si $x\varepsilon N_1$, noto valore de $f\mathrm{x}^{-1}$ pro omni numero de classe $1\cdots x$, valore de $f\mathrm{x}^{-1}(x+1)$ resulta ex valores de f correspondente ad numeros de classe $1\cdots x$ per resolutione de æquatione de gradu 1, ubi coefficiente de incognita es $f1$. Ergo nos pone

❋ 10. $f\varepsilon\, \mathrm{q}'FN_1 \,.\, f1 \neq 0 \,.\supset.$

$\cdot 0 \quad f\mathrm{x}^{-1} = \imath[\mathrm{q}'FN_1 \cap h\ni(f\mathrm{x}h=a)]$ Df

$\cdot 1 \quad f\mathrm{x}^{-1}\,\varepsilon\, \mathrm{q}'FN_1 \,.\, f\mathrm{x}(f\mathrm{x}^{-1}) = a$

Denominatione de functione coniugato es adoptato a Cesàro.

Calculo de valore de functione coniugato de f pote fi aut per valore de cointegrale de f, per P sequente, que nos obtine ex $(f\mathrm{x}^{-1}\,|\,g)$P9·1:

$\cdot 2 \quad n,m\varepsilon N_1 \,.\, n>\tau m \,.\supset.$
$f\mathrm{x}^{-1}m = \Sigma[(-1)^{r-1}\times C(n,r)\times f\mathrm{x}^{r-1}m\,/(f1)^{r-1}\,|r, N_1]$

aut per valore de cointegrales de $f-(f1)\times a$, per P sequente

$\cdot 3 \quad \mathrm{Hp}\cdot 2 \,.\supset.\, f\mathrm{x}^{-1}m = \Sigma[(-1)^r\times(f-(f1)\times a)\mathrm{x}^r m\,/(f1)^{r+1}\,|r, N_0]$

ubi omni termine de membro 2 fi nullo quando r supera τm. Ut demonstra ce P nos pone

$$h\varepsilon\, \mathrm{q}'FN_1 \,.\, hm = \Sigma[(-1)^r\times(f-(f1)\times a)\mathrm{x}^r m\,/(f1)^{r+1}\,|r, N_0]$$

et nos fac producto integrale de duo membro de æqualitate præcedente per fm, posito sub forma $(f1)\times am+(f-(f1)\times a)m$. Nos obtine ex Distrib(x,+):

$(h\mathrm{x}f)m = (h\mathrm{x}(f1)\times a)m + \Sigma[(-1)^r\times(f-(f1)\times a)\mathrm{x}^{r+1}m\,/(f1)^{r+1}\,|r, N_0]$
$= (hm)\times(f1) - (f1)\times\Sigma[(-1)^{r+1}\times(f-(f1)\times a)\mathrm{x}^{r+1}m/(f1)^{r+1}\,|r, N_0]$
$= (hm)\times(f1) - (f1)(hm-(am)/(f1)) = am$

Ergo es $hm = f\mathrm{x}^{-1}m$.

Multo utile pro calculo de valore de functione coniugato de functione imprimitivo es P sequente

$\cdot 4 \quad f\varepsilon\, (\mathrm{q}'FN_1)\mathrm{impr} \,.\supset.\, f\mathrm{x}^{-1}\,\varepsilon\,(\mathrm{q}'FN_1)\mathrm{impr}$

Nos demonstra illo in modo indirecto et multo simplice.

In vero si nos pone

$h = \imath\,(\mathrm{q}'FN_1)\mathrm{impr}\cap k\ni[m\varepsilon N_1 \,.\, p\varepsilon \mathrm{Np} \,.\supset_{m,n}.\, k(p^m) = f\mathrm{x}^{-1}(p^m)]$ (1)

nos subito trahe, si $x\varepsilon N_1$,

$(f\mathrm{x}h)x = \Pi[(f\mathrm{x}h)(p\upharpoonright \mathrm{mp}(p,x))\,|p, \mathrm{Np}\cap x/N_1]$ (2)

$(2)\,.\,\mathrm{Hp}(1) \,.\supset.\, (f\mathrm{x}h)x = \Pi[(f\mathrm{x}f\mathrm{x}^{-1})(p\upharpoonright \mathrm{mp}(p,x))\,|p, \mathrm{Np}\cap x/N_1]$
$= \Pi[a(p\upharpoonright \mathrm{mp}(p,x))\,|p, \mathrm{Np}_1\cap x/N_1] = ax$

Ergo $h = f\mathrm{x}^{-1}$.

Sed, ex Hp(1), h es functione imprimitivo, ergo et $f\mathrm{x}^{-1}$ es functione imprimitivo.

In basi ad ce P calculo de valore de functione coniugato de functione imprimitivo es reducto ad calculo de valores que functione sume pro potestate de numero primo.

Nos nunc defini cointegrale de gradu integro et negativo:

$\cdot 5 \quad s \varepsilon \mathrm{N}_1 . \supset . f\mathrm{x}^{-s} = (f\mathrm{x}^{-1})\mathrm{x}^s$ Df

Es facile extende ad cointegrale de gradu integro et negativo proprietate de cointegrale de gradu integro et positivo, id es P8·1·2·3. Nos da postea et algorithmo pro calculo directo de cointegrale da gradu integro et negativo.

$\cdot 6 \quad g \varepsilon \mathrm{q'FN}_1 . \supset . g\mathrm{x}f\mathrm{x}^{-1} = \iota(\mathrm{q'FN}_1) \cap k \ni (g = f\mathrm{x}k)$

Dem.

Hp . Assoc x $. \supset . f\mathrm{x}(g\mathrm{x}f\mathrm{x}^{-1}) = (f\mathrm{x}f\mathrm{x}^{-1})\mathrm{x}g = \alpha\mathrm{x}g = g$ (1)

Hp . (1) . P1·7 $. \supset .$ P.

Nos appella $g\mathrm{x}f\mathrm{x}^{-1}$ « quoto integrale de g per f ».

§ 7. Functione μ de Möbius

❋ 11·0 $\quad \mu = v\mathrm{x}^{-1}$ Df

Ex P10·4 seque

$\cdot 1 \quad \mu \ \varepsilon \ (\mathrm{q'FN}_1)\mathrm{impr}$

Ergo nos pote deduce valore de μ

$\cdot 2 \quad \mu 1 = 1$ Dfp

$\cdot 21 \quad p \varepsilon \mathrm{Np} . m \varepsilon \mathrm{N}_1 + 1 . \supset . \mu p = -1 . \mu(p^m) = 0$ Dfp

$\cdot 22 \quad n \varepsilon \mathrm{N}_1 - (\mathrm{N}_1 \times \mathrm{N}_1^2) . \supset . \mu n = (-1) \lceil (\varkappa n)$ Dfp

$\cdot 23 \quad n \varepsilon (\mathrm{N}_1 \times \mathrm{N}_1^2) - \iota 1 . \supset . \mu n = 0$ Dfp

Functione μ es appellato « functione de MŒBIUS » vel « de MERTENS » (v. MŒBIUS, J. f. Math, t.9, a.1832 p.105 = Werke, t.4, p.589; MERTENS, J. f. Math. t.77, a.1874, p.289).

$\cdot 3 \quad \int \mu = \alpha$

Dm. P11·0 $. \supset . \int \mu = \mu\mathrm{x}v = (v\mathrm{x}^{-1})\mathrm{x}v = \alpha$

$\cdot 31 \quad \mu = \Sigma[(-1)^r \times (v - \alpha)\mathrm{x}^r | r, \mathrm{N}_0]$

Dm. P·0 . P10·3 $. \supset .$ P

Propositione notabile pro calculo de coniugato de functione composito es sequente:

$\cdot 4 \quad f \varepsilon (\mathrm{q'FN}_1)\mathrm{comp} . \supset . f\mathrm{x}^{-1} = \mu \times f$

Dm. Hp . P2·3 $. \supset . f\mathrm{x}(\mu \times f) = f \times (v\mathrm{x}\mu) = f \times \alpha = \alpha . \supset .$ P

Propositione præcedente es casu particulare de hoc theorema, que habe analogo demonstratione:

$\cdot 5 \quad f \varepsilon (\mathrm{q'FN}_1)\mathrm{comp} . g \varepsilon \mathrm{q'FN}_1 . \supset . (f \times g)\mathrm{x}^{-1} = f \times g\mathrm{x}^{-1}$

$\cdot 6 \quad a \varepsilon \mathrm{N}_1 : x \varepsilon \mathrm{N}_1 . \supset_x . fx = a .\because. \supset . f\mathrm{x}^{-1} = \mu/a$

Dm. Hp . P5·2 $. \supset . f = a \times v$ (1)

(1) $. \supset . f\mathrm{x}\mu/a = (a \times v)\mathrm{x}(\mu/a) = v\mathrm{x}\mu = \alpha . \supset .$ P

§ 8 Derivata numerico

Nunc nos considera producto integrale de functione numerico f per functione μ, que nos, simul cum BOUGAÏEFF et CESÀRO, appella « derivata numerico de f » et indica cum symbolo ∂f.

❋ 12. $\quad f \varepsilon \mathrm{q'FN}_1 . \supset .$

$\cdot 0 \quad \partial f = f\mathrm{x}\mu$ Df

$\cdot 1 \quad \int \partial f = \partial \int f = f$

Ce P es appellato « lege de inversione de integrale numerico ». Si functione f es integrale numerico de functione numerico incognito g, g es derivata numerico de f; ergo valore de g es deducto per calculo de producto integrale $f\mathrm{x}\mu$, et nos habe P sequente (P·2·22·23)

$\cdot 2 \quad x \varepsilon \mathrm{N}_1 . \supset . \partial fx =$
$fx + \Sigma\{[(-1) \lceil (\varkappa a)] \times f(x/a) \mid a, (\mathrm{N}_1 \cap x/\mathrm{N}_1) - (\mathrm{N}_1 \times \mathrm{N}_1^2)\}$

Ce P es dato quasi in idem tempus a DEDEKIND (J. für Math., t.54, a.1857, p.1) et a LIOUVILLE (J, de Math., s.2, t.2, a.1857, pag.110).

·3 $f \varepsilon (\mathrm{q}'\mathrm{FN}_1)\mathrm{impr} . \supset . \partial f \, \varepsilon \, (\mathrm{q}'\mathrm{FN}_1)\mathrm{impr}$

In phrasi: derivata numerico de functione imprimitivo es et functione numerico imprimitivo. Ce P seque ex P4·4, 11·1.
Ergo calculo de derivata numerico de functione imprimitivo pote fi per P sequente

·4 $\mathrm{Hp}\cdot 3 \, . \, x \varepsilon \mathrm{N}_1 + 1 . \supset .$
$\partial f x = \Pi \{ f(a \upharpoonright \mathrm{mp}(a,x)) - f(a \upharpoonright [\mathrm{mp}(a,x) - 1]) | x, \mathrm{Np} \frown x/\mathrm{N}_1 \}$

In particulare

·41 $\mathrm{Hp}\cdot 4 \, . \, f \varepsilon \, (\mathrm{q}'\mathrm{FN}_1)\mathrm{comp} . \supset .$
$\partial f x = \Pi \{ f(a \upharpoonright [\mathrm{mp}(a,x) - 1]) \times (fa - 1) | a, \mathrm{Np} \frown x/\mathrm{N}_1 \}$

Exemplo: si $x \varepsilon \mathrm{N}_1$, tunc $x|x \; \varepsilon \; (\mathrm{q}'\mathrm{FN}_1)\mathrm{comp}$ et per P·3 $\partial(x|x) \; \varepsilon \; (\mathrm{q}'\mathrm{FN}_1)\mathrm{impr}$, et ex P·41 seque

·42 $x \varepsilon \mathrm{N}_1 . \supset . \partial(x|x) = \{ x \times \Pi[(1 - /a), \mathrm{Np} \frown x/\mathrm{N}_1] \} | x$
$\partial(x|x)$ es indicato a GAUSS cum symbolo Φ.

Pro historia de Φ vide Form. 1905, III, §18. P·42 = Form. 1905, III, §16, P·6.

·43 $\Phi = \partial(x|x)$

✻ 13. $f \varepsilon \, \mathrm{q}'\mathrm{FN}_1 \, . \, r \varepsilon \mathrm{N}_0 . \supset .$

·0 $\partial^0 f = f \, . \, \partial^{r+1} f = \partial(\partial^r f)$ Df

Lege $\partial^r f$ « derivata numerico de f de ordine r ».

·1 $\partial^r f = f \mathrm{x} (\mu \mathrm{x}^r)$

Pro calculo de derivata numerico de ordine f de functione imprimitivo es utile P sequente:

·2 $a \varepsilon \mathrm{Np} \, . \, n \varepsilon \mathrm{N}_1 . \supset . \partial^r f(a^n) = \Sigma[(-1)^s \times \mathrm{C}(r,s) \times f(a^{n-s}) | s, 0 \cdots n]$

Deductione de ce P ex P12·4 per lege de inductione es facile.
In casu particulare:

·3 $x \varepsilon \mathrm{N}_1 + 1 . \supset . \partial^r \mu x = [(-1) \upharpoonright (\tau x)] \times \Pi[\mathrm{C}(r+1, \mathrm{mp}(s,x)) | s, \mathrm{Np}]$

Pro calculo de derivata numerico de functione logarithmico es utile P sequente

·4 $l \varepsilon \, (\mathrm{q}'\mathrm{FN}_1)\log \, . \, x \varepsilon \mathrm{N}_1 + 1 . \supset .$
$\partial^r l x = \Sigma \{ \mu \mathrm{x}^{r-1} (x/[a \upharpoonright \mathrm{mp}(a,x)] \times \partial^r l(a \upharpoonright \mathrm{mp}(a,x)) | a, \mathrm{Np} \frown x/\mathrm{N}_1 \}$
Dm. $\mathrm{Hp} \, . \, (\mu \mathrm{x}^r | f)\mathrm{P}6\cdot 6 . \supset . \mathrm{P}$

·5 $\mathrm{Hp}\cdot 4 \, . \, x \varepsilon (\mathrm{N}_1 + 1) - (\mathrm{Np} \upharpoonright \mathrm{N}_1) . \supset . \partial l x = 0$
Dm. $\mathrm{Hp}. \, (1|r)\mathrm{P}\cdot 4 . \supset . \mu \mathrm{x}^0 = \alpha . \supset . \mathrm{P}$

·6 $f1 -= 0 . \supset . (\partial^r f)\mathrm{x}^{-1} = \int^r (f \mathrm{x}^{-1})$
Dm. $\mathrm{Hp} . \supset . \int^r (f \mathrm{x}^{-1}) \mathrm{x} \partial^r f) = (f \mathrm{x}^{-1}) \mathrm{x}) (v \mathrm{x}^r) \, \mathrm{x} \, (f \mathrm{x})(v \mathrm{x}^r)) =$
$= (f \mathrm{x}^{-1})(\mathrm{x} f)) \mathrm{x} (v \mathrm{x} \mu) \mathrm{x}^r = \alpha \mathrm{x} (\alpha \mathrm{x}^r) = \alpha . \supset . \mathrm{P}$

In modo analogo:

·7 $\mathrm{Hp}\cdot 6 . \supset . (\int^r f)\mathrm{x}^{-1} = \partial^r (f \mathrm{x}^{-1})$

·8 $\mathrm{Hp}\cdot 6 . \supset . f \mathrm{x}^{-1} = \int^r [(\int^r f)\mathrm{x}^{-1}]$
Dm. $(\int^r f | f)\mathrm{P}\cdot 6 . \supset . \mathrm{P}$

·81 $f \mathrm{x}^{-1} = \partial^r [(\partial^r f)\mathrm{x}^{-1}]$
Dm. $(\partial^r f | f)\mathrm{P}\cdot 6 . \supset . \mathrm{P}$

P8·81 es utile pro calculo de functione coniugato de f quando es noto functione coniugato aut de $\int^r f$ aut de $\partial^r f$.

In particulare seque:

·82 $\int^r f \; \varepsilon \; (\mathrm{q}'\mathrm{FN}_1)\mathrm{comp} . \supset . f \mathrm{x}^{-1} = \int^r (\mu \times \int^r f)$

·83 $\partial^r f \; \varepsilon \; (\mathrm{q}'\mathrm{FN}_1)\mathrm{comp} . \supset . f \mathrm{x}^{-1} = \partial^r (\mu \times \partial^r f)$

Pro calculo de membro 2 in æqualitate ·82 ·83 es utile to observa que functione in tali membro es imprimitivo.

Ut applicatione de P præcedente nos inveni P sequente:

·84 $x \varepsilon \, \mathrm{N}_1 + 1 . \supset . \Phi \mathrm{x}^{-1} x = [(-1) \upharpoonright (\varkappa x)] \times \Pi[(1 - a) | a, \mathrm{Np} \frown x/\mathrm{N}_1]$

·85 » $. \supset . \sigma \mathrm{x}^{-1} x = [(-1) \upharpoonright (\tau x)] \times \Pi[\mathrm{C}(1, \mathrm{mp}(a,x)) -$
$a \times \mathrm{C}(1, \mathrm{mp}(a,x) - 1) \, | a, \mathrm{Np} \frown x/\mathrm{N}_1]$

·9 $x \varepsilon \mathrm{N}_1 \, . \, m \varepsilon \, \mathrm{N}_1 + \tau x . \supset .$
$f x = \Sigma[(-1)^{s-1} \times \mathrm{C}(m,s) \times \partial^s f x | s, \mathrm{N}_1]$
Dm. $(\mu | f, f | g)\mathrm{P}9\cdot 1 . \supset . \mathrm{P}$

§ 9. Factoriale integrale

✻ 14. $f \varepsilon\, (q' - \iota 0)FN_1 \,.\, g \varepsilon\, q'FN_1 \,.\, x \varepsilon N_1 \,.\supset.$

$\cdot 0 \quad P(f,g,x) = \Pi[f(x/r) \upharpoonright (gr) | r,\ N_1 \cap x/N_1]$ Df

$\cdot 01 \quad P(f,g) = P(f,g,x)|x$ Df

Lege $P(f,g,x)$ « factoriale integrale de f ad g secundo x ».

$\cdot 02 \quad P(f,a) = f$

$\cdot 1 \quad \log P(f,g) = \log f \times g$

Ce P seque ex Df ·0, quando nos sume logarithmo de duo membro. Ergo nos pote deduce proprietate de factoriale integrale ex proprietate de producto integrale.

$\cdot 2 \quad f,g\ \varepsilon\ (q'FN_1)\mathrm{impr} \,.\supset.\ P(f,g,x) =$
$\Pi[\ P[f,\ g,\ a \upharpoonright mp(a,x)] \upharpoonright \int g \} x \,/\, [a \upharpoonright mp(a,x)] \} \ |a,\ Np \cap x/N_1\ ]$

Dm. $\quad Hp \,.\supset.\ \log f = (q'FN_1)\log$ (1)

$(1) \,.\, P\cdot 1 \,.\, (g | f,\ \log f | l)\, P6\cdot 6 \,.\supset.\ \log P(f,g,x) =$
$\Sigma \} [(\log f \times g)(a \upharpoonright mp(a,x))] \times \int g(x \,/\, [a \upharpoonright mp(a,x)])\ |a,\ Np \}$ (2)

$Hp \,.\, a \varepsilon Np \,.\, (a \upharpoonright mp(a,x)\ |\ x) P\cdot 1 \,.\supset.\ \log P(f,g,\ a \upharpoonright mp(a,x)) =$
$(\log f \times g)(a \upharpoonright mp(a,x))$ (3)

$(2) \,.\, (3) \,.\supset.\ \log P(f,g,x) = \Sigma[\log P(f,g,a \upharpoonright mp(a,x)) \times$
$\int g(x \,/\, [a \upharpoonright mp(a,x)])\ |a,\ Np \cap x/N_1] =$
$\log \Pi | \} P(f,\ g,\ a \upharpoonright mp(a,x) \} \upharpoonright \int g(x \,/\, [a \upharpoonright mp(a,x)])\ |a,\ Np \cap x/N_1]$ (4)

$(4) \,.\supset.\ P.$

$\cdot 3 \quad P(f,v,x) = \Pi(f, N_1 \cap x/N_1)$

Ce P seque subito ex Df ·0. Nos lege $P(f,v)$ « factoriale integrale de f ».

Ex P·2 seque:

$\cdot 4 \quad P(x|x,\ v,x) = \Pi(N_1 \cap x/N_1) = x \upharpoonright (\nu x/2)$

$\cdot 5 \quad P(\nu,v,x) = \Pi[\} (mp(a,x)+1)! \} \upharpoonright \nu(x/[a \upharpoonright mp(a,x)])\ |a,\ Np]$

$\cdot 6 \quad P(\Phi,v,x) = [x \upharpoonright (\nu x/2)] \times \Pi \} (1 - /a) \upharpoonright (mp(a,x) \times$
$\nu(x/a[mp(a,x)]) | a,\ Np \}$

$\cdot 7 \quad f \varepsilon\ (q'FN_1)\mathrm{impr} \,.\, x \varepsilon\ N_1 - (Np \upharpoonright N_1) \,.\supset.\ P(f,\mu,x) = 1$

✻ 15. $f \varepsilon\, (q' - \iota 0)FN_1 \,.\, g \varepsilon\, q'FN_1 \,.\, x \varepsilon N_1 \,.\supset:$

$\cdot 0 \quad h \varepsilon\, (q' - \iota 0)FN_1 \,.\, k = P(f,\ g \times h\mathrm{x}^{-1}) \,.\supset.$
$P(f,g) = P(k,h)$

Ce P es notabile, quare per ipso nos pote exprime factoriale integrale de f ad g cum alio factoriale integrale in que secundo functione numerico es arbitrario.

Dm. $\quad Hp \,.\, P\cdot 1 \,.\supset.\ \log P(k,h) = \log k \times h$ (1)

$Hp \,.\supset.\ \log k = \log f \times (g \times h\mathrm{x}^{-1})$ (2)

$(2) \,.\, \mathrm{Assoc}\ \times \,.\supset.\ \log k = \log f \times g \times h\mathrm{x}^{-1}$ (3)

$(1) \,.\, (3) \,.\, \mathrm{Assoc}\ \times \,.\supset.\ \log P(k,h) = \log f \times g \times h \times h\mathrm{x}^{-1} = \log f \times g$ (4)

$(4) \,.\, P\cdot 1 \,.\supset.\ \log P(h,k) = \log P(f,g) \,.\supset.\ P$

Ex ce P. nos deduce duo consequentia notabile:

$\cdot 1 \quad k = P(f,\partial g) \,.\supset.\ P(f,g) = P(k,v)$

In phrasi: omni factoriale integrale de uno functione numerico ad alio functione numerico pote es reducto ad simplice factoriale integrale.

Dm. $(v|h)P\cdot 0 \,.\supset.\ P$

$\cdot 2 \quad g = P(f,v) \,.\supset.\ f = P(g,\mu)$

In phrasi: si g es factoriale integrale de f, tunc f es factoriale integrale de f ad μ.

Dm. $(a|g,g|f)P\cdot 0 \,.\, P\cdot 02 \,.\supset.\ P$

§ 10. Functione analytico numerico-integrale

✻ 16. $a \varepsilon\, q'FN_0 \,.\supset.$

$\cdot 0 \quad \mathrm{radio}\ a = l' \bmod \{ q' \cap z \ni [\Sigma(a_s z^s\ |s,\ N_0)\ \varepsilon\ q'] \}$ Df

Lege radio a « radio de convergentia de serie $\Sigma(a_s z^s\ |s,\ N_0)$, pro z ».

$\cdot 1 \quad f \varepsilon\ q'FN_1 \,.\, r = \mathrm{radio}\ a \,.\, n \varepsilon N_1 : d \varepsilon\ N_1 \cap n/N_1 \,.\supset_d.\ \bmod(fd) < r$
$:\supset.\ \Sigma[\bmod(a_s \times f\mathrm{x}^s n)\ |s,\ N_0]\ \varepsilon\ q'$

In phrasi: Si r es radio de convergentia de serie . . . . . . $\Sigma(a_s z^s | s, N_0)$ et si modulo de valore que functione numerico f sume pro omni divisore de numero integro et positivo n es

·3 $f\varepsilon(\mathrm{q'FN_1})\mathrm{impr} \,.\supset.\, \partial f \,\varepsilon\, (\mathrm{q'FN_1})\mathrm{impr}$

In phrasi: derivata numerico de functione imprimitivo es et functione numerico imprimitivo. Ce P seque ex P4·4, 11·1. Ergo calculo de derivata numerico de functione imprimitivo pote fi per P sequente

·4 $\mathrm{Hp}\cdot 3 \,.\, x\varepsilon \mathrm{N_1}+1 \,.\supset.$
$\partial fx = \Pi\{f(a\upharpoonright\mathrm{mp}(a,x)) - f(a\upharpoonright[\mathrm{mp}(a,x)-1])|x,\ \mathrm{Np}\frown x/\mathrm{N_1}\}$

In particulare

·41 $\mathrm{Hp}\cdot 4 \,.\, f\varepsilon\, (\mathrm{q'FN_1})\mathrm{comp} \,.\supset.$
$\partial fx = \Pi\{f(a\upharpoonright[\mathrm{mp}(a,x)-1])\times(fa-1)|a,\ \mathrm{Np}\frown x/\mathrm{N_1}\}$

Exemplo: si $x\varepsilon\mathrm{N_1}$, tunc $x|x \,\varepsilon\, (\mathrm{q'FN_1})\mathrm{comp}$ et per P·3 $\partial(x|x) \,\varepsilon\, (\mathrm{q'FN_1})\mathrm{impr}$, et ex P·41 seque

·42 $x\varepsilon\mathrm{N_1} \,.\supset.\, \partial(x|x) = \{x\times\Pi[(1-/a),\ \mathrm{Np}\frown x/\mathrm{N_1}]\}|x$
$\partial(x|x)$ es indicato a GAUSS cum symbolo Φ.

Pro historia de Φ vide Form. 1905, III, §18. P·42 = Form. 1905, III, §16, P·6.

·43 $\Phi = \partial(x|x)$

✻ 13. $f\varepsilon\, \mathrm{q'FN_1} \,.\, r\varepsilon\mathrm{N_0} \,.\supset.$

·0 $\partial^0 f = f \,.\, \partial^{r+1}f = \partial(\partial^r f)$ Df

Lege $\partial^r f$ « derivata numerico de f de ordine r ».

·1 $\partial^r f = f\mathrm{x}(\mu\mathrm{x}^r)$

Pro calculo de derivata numerico de ordine f de functione imprimitivo es utile P sequente:

·2 $a\varepsilon\mathrm{Np} \,.\, n\varepsilon\mathrm{N_1} \,.\supset.\, \partial^r f(a^n) = \Sigma[(-1)^s\times \mathrm{C}(r,s)\times f(a^{n-s})|s,\ 0\cdots n]$

Deductione de ce P ex P12·4 per lege de inductione es facile. In casu particulare:

·3 $x\varepsilon\mathrm{N_1}+1 \,.\supset.\, \partial^r \mu x = [(-1)\upharpoonright(\tau x)]\times\Pi[\mathrm{C}(r+1,\ \mathrm{mp}(s,x))|s,\ \mathrm{Np}]$

Pro calculo de derivata numerico de functione logarithmico es utile P sequente

·4 $l\varepsilon\, (\mathrm{q'FN_1})\log \,.\, x\varepsilon\mathrm{N_1}+1 \,.\supset.$
$\partial^r lx = \Sigma\{\mu\mathrm{x}^{r-1}(x/[a\upharpoonright\mathrm{mp}(a,x)]\times\partial^r l(a\upharpoonright\mathrm{mp}(a,x))|a,\ \mathrm{Np}\frown x/\mathrm{N_1}\}$
Dm. $\mathrm{Hp} \,.\, (\mu\mathrm{x}^r|f)\mathrm{P}6\cdot 6 \,.\supset.\, \mathrm{P}$

·5 $\mathrm{Hp}\cdot 4 \,.\, x\varepsilon(\mathrm{N_1}+1)-(\mathrm{Np}\upharpoonright\mathrm{N_1}) \,.\supset.\, \partial lx = 0$
Dm. $\mathrm{Hp}.\ (1|r)\mathrm{P}\cdot 4 \,.\supset.\, \mu\mathrm{x}^0 = \alpha \,.\supset.\, \mathrm{P}$

·6 $f1 -= 0 \,.\supset.\, (\partial^r f)\mathrm{x}^{-1} = \int^r(f\mathrm{x}^{-1})$
Dm. $\mathrm{Hp} \,.\supset.\, \int^r(f\mathrm{x}^{-1})\mathrm{x}\partial^r f) = (f\mathrm{x}^{-1})\mathrm{x})(v\mathrm{x}^r)\,\mathrm{x}\,(f\mathrm{x})(v\mathrm{x}^r)) =$
$= (f\mathrm{x}^{-1})(\mathrm{x}f))\mathrm{x}(v\mathrm{x}\mu)\mathrm{x}^r = \alpha\mathrm{x}(\alpha\mathrm{x}^r) = \alpha \,.\supset.\, \mathrm{P}$

In modo analogo:

·7 $\mathrm{Hp}\cdot 6 \,.\supset.\, (\int^r f)\mathrm{x}^{-1} = \partial^r(f\mathrm{x}^{-1})$

·8 $\mathrm{Hp}\cdot 6 \,.\supset.\, f\mathrm{x}^{-1} = \int^r[(\int^r f)\mathrm{x}^{-1}]$
Dm. $(\int^r f|f)\mathrm{P}\cdot 6 \,.\supset.\, \mathrm{P}$

·81 $f\mathrm{x}^{-1} = \partial^r[(\partial^r f)\mathrm{x}^{-1}]$
Dm. $(\partial^r f|f)\mathrm{P}\cdot 6 \,.\supset.\, \mathrm{P}$

P8·81 es utile pro calculo de functione coniugato de f quando es noto functione coniugato aut de $\int^r f$ aut de $\partial^r f$.

In particulare seque:

·82 $\int^r f \,\varepsilon\, (\mathrm{q'FN_1})\mathrm{comp} \,.\supset.\, f\mathrm{x}^{-1} = \int^r(\mu\times\int^r f)$

·83 $\partial^r f \,\varepsilon\, (\mathrm{q'FN_1})\mathrm{comp} \,.\supset.\, f\mathrm{x}^{-1} = \partial^r(\mu\times\partial^r f)$

Pro calculo de membro 2 in æqualitate ·82 ·83 es utile to observa que functione in tali membro es imprimitivo.

Ut applicatione de P præcedente nos inveni P sequente:

·84 $x\varepsilon\ \mathrm{N_1}+1 \,.\supset.\, \Phi\mathrm{x}^{-1}x = [(-1)\upharpoonright(\varkappa x)]\times\Pi[(1-a)|a,\ \mathrm{Np}\frown x/\mathrm{N_1}]$

·85 » $.\supset.\, \sigma\mathrm{x}^{-1}x = [(-1)\upharpoonright(\tau x)]\times\Pi[\mathrm{C}(1,\mathrm{mp}(a,x)) -$
$a\times\mathrm{C}(1,\mathrm{mp}(a,x)-1)\ |a,\ \mathrm{Np}\frown x/\mathrm{N_1}]$

·9 $x\varepsilon\mathrm{N_1} \,.\, m\varepsilon\ \mathrm{N_1}+\tau x \,.\supset.$
$fx = \Sigma[(-1)^{s-1}\times\mathrm{C}(m,s)\times\partial^s fx|s,\ \mathrm{N_1}]$
Dm. $(\mu|f,\ f|g)\mathrm{P}9\cdot 1 \,.\supset.\, \mathrm{P}$

§ 9. **Factoriale integrale**

❋ 14. $f\varepsilon\,(\mathrm{q}'\text{-}\iota 0)\mathrm{FN}_1 \,.\, g\varepsilon\,\mathrm{q}'\mathrm{FN}_1 \,.\, x\varepsilon\mathrm{N}_1 \,.\supset.$

·0 $\mathrm{P}(f,g,x) = \mathit{\Pi}\{f(x/r)\upharpoonright(gr)|r,\ \mathrm{N}_1\cap x/\mathrm{N}_1\}$ Df

·01 $\mathrm{P}(f,g) = \mathrm{P}(f,g,x)|x$ Df

Lege $\mathrm{P}(f,g,x)$ « factoriale integrale de f ad g secundo x ».

·02 $\mathrm{P}(f,a) = f$

·1 $\log\mathrm{P}(f,g) = \log f \times g$

Ce P seque ex Df ·0, quando nos sume logarithmo de duo membro. Ergo nos pote deduce proprietate de factoriale integrale ex proprietate de producto integrale.

·2 $f,g\ \varepsilon\,(\mathrm{q}'\mathrm{FN}_1)\mathrm{impr} \,.\supset.\, \mathrm{P}(f,g,x) =$
$\mathit{\Pi}[\,\mathrm{P}[f,\, g,\, a\upharpoonright\mathrm{mp}(a,x)] \upharpoonright \int g\{x \,/\, [a\upharpoonright\mathrm{mp}(a,x)]\}\ |a,\ \mathrm{Np}\cap x/\mathrm{N}_1\,]$

Dm. Hp $.\supset.$ $\log f = (\mathrm{q}'\mathrm{FN}_1)\log$ (1)
(1) . P·1 . $(g\,|\,f,\ \log f\,|\,l)$ P6·6 $.\supset.$ $\log\mathrm{P}(f,g,x) =$
$\Sigma\{[(\log f \times g)(a\upharpoonright\mathrm{mp}(a,x))] \times \int g(x\,/\,[a\upharpoonright\mathrm{mp}(a,x)])\ |a,\ \mathrm{Np}\}$ (2)
Hp . $a\varepsilon\mathrm{Np}$. $(a\upharpoonright\mathrm{mp}(a,x)\ |\ x)$P·1 $.\supset.$ $\log\mathrm{P}(f,g,\ a\upharpoonright\mathrm{mp}(a,x)) =$
$(\log f \times g)(a\upharpoonright\mathrm{mp}(a,x))$ (3)
(2) . (3) $.\supset.$ $\log\mathrm{P}(f,g,x) = \Sigma[\log\mathrm{P}(f,g,a\upharpoonright\mathrm{mp}(a,x)) \times$
$\int g(x\,/\,[a\upharpoonright\mathrm{mp}(a,x)])\ |a,\ \mathrm{Np}\cap x/\mathrm{N}_1] =$
$\log \mathit{\Pi}[\{\mathrm{P}(f,\, g,\, a\upharpoonright\mathrm{mp}(a,x)\}\upharpoonright \int g(x\,/\,[a\upharpoonright\mathrm{mp}(a,x)])\ |a,\ \mathrm{Np}\cap x/\mathrm{N}_1]$ (4)
(4) $.\supset.$ P.

·3 $\mathrm{P}(f,v,x) = \mathit{\Pi}(f,\mathrm{N}_1\cap x/\mathrm{N}_1)$

Ce P seque subito ex Df·0. Nos lege $\mathrm{P}(f,v)$ « factoriale integrale de f ».
Ex P·2 seque:

·4 $\mathrm{P}(x|x,\ v,x) = \mathit{\Pi}(\mathrm{N}_1\cap x/\mathrm{N}_1) = x\upharpoonright(\nu x/2)$

·5 $\mathrm{P}(\nu,v,x) = \mathit{\Pi}[\{(\mathrm{mp}(a,x)+1)!\}\upharpoonright \nu(x/[a\upharpoonright\mathrm{mp}(a,x)])\ |a,\ \mathrm{Np}]$

·6 $\mathrm{P}(\Phi,v,x) = [x\upharpoonright(\nu x/2)] \times \mathit{\Pi}\{(1-/a)\upharpoonright(\mathrm{mp}(a,x) \times$
$\nu(x/a[\mathrm{mp}(a,x)])\}|a,\ \mathrm{Np}\}$

·7 $f\varepsilon\,(\mathrm{q}'\mathrm{FN}_1)\mathrm{impr} \,.\, x\varepsilon\,\mathrm{N}_1\text{-}(\mathrm{Np}\upharpoonright\mathrm{N}_1) \,.\supset.\, \mathrm{P}(f,\mu,x) = 1$

❋ 15. $f\varepsilon\,(\mathrm{q}'\text{-}\iota 0)\mathrm{FN}_1 \,.\, g\varepsilon\,\mathrm{q}'\mathrm{FN}_1 \,.\, x\varepsilon\mathrm{N}_1 \,.\supset:$

·0 $h\varepsilon\,(\mathrm{q}'\text{-}\iota 0)\mathrm{FN}_1 \,.\, k = \mathrm{P}(f,\ g \times h\mathrm{x}^{-1}) \,.\supset.$
$\mathrm{P}(f,g) = \mathrm{P}(k,h)$

Ce P es notabile, quare per ipso nos pote exprime factoriale integrale de f ad g cum alio factoriale integrale in que secundo functione numerico es arbitrario.

Dm. Hp . P·1 $.\supset.$ $\log\mathrm{P}(k,h) = \log k \times h$ (1)
Hp $.\supset.$ $\log k = \log f \times (g \times h\mathrm{x}^{-1})$ (2)
(2) . Assoc x $.\supset.$ $\log k = \log f \times g \times h\mathrm{x}^{-1}$ (3)
(1) . (3) . Assoc x $.\supset.$ $\log\mathrm{P}(k,h) = \log f \times g \times h \times h\mathrm{x}^{-1} = \log f \times g$ (4)
(4) . P·1 $.\supset.$ $\log\mathrm{P}(h,k) = \log\mathrm{P}(f,g)$ $.\supset.$ P

Ex ce P. nos deduce duo consequentia notabile:

·1 $k = \mathrm{P}(f,\partial g) \,.\supset.\, \mathrm{P}(f,g) = \mathrm{P}(k,v)$

In phrasi: omni factoriale integrale de uno functione numerico ad alio functione numerico pote es reducto ad simplice factoriale integrale.
Dm. $(v|h)$P·0 $.\supset.$ P

·2 $g = \mathrm{P}(f,v) \,.\supset.\, f = \mathrm{P}(g,\mu)$

In phrasi: si g es factoriale integrale de f, tunc f es factoriale integrale de f ad μ.
Dm. $(a|g,g|f)$P·0 . P·02 $.\supset.$ P

§ 10. **Functione analytico numerico-integrale**

❋ 16. $a\varepsilon\,\mathrm{q}'\mathrm{FN}_0 \,.\supset.$

·0 radio a = $\mathrm{l}'\ \mathrm{mod}\,\{\mathrm{q}'\cap z\ni[\Sigma(a_s z^s\,|s,\ \mathrm{N}_0)\ \varepsilon\ \mathrm{q}']\}$ Df

Lege radio a « radio de convergentia de serie $\Sigma(a_s z^s\,|s,\ \mathrm{N}_0)$, pro z ».

·1 $f\varepsilon\,\mathrm{q}'\mathrm{FN}_1 \,.\, r = \text{radio } a \,.\, n\varepsilon\mathrm{N}_1 : d\varepsilon\ \mathrm{N}_1\cap n/\mathrm{N}_1 \,.\supset_d.\ \mathrm{mod}(fd) < r$
$:\supset.\ \Sigma[\mathrm{mod}(a_s \times f\mathrm{x}^s n)\ |s,\ \mathrm{N}_0]\ \varepsilon\ \mathrm{q}'$

In phrasi: Si r es radio de convergentia de serie . . . . . . $\Sigma(a_s z^s|s,\mathrm{N}_0)$ et si modulo de valore que functione numerico f sume pro omni divisore de numero integro et positivo n es minore de r, tunc serie $\Sigma(a_s \times f\mathrm{x}^s n \,|s, \mathrm{N}_0)$ es convergente in modo absoluto.

In vero si m es maximo ex valores que mod$f$ sume pro omni divisore de n, ex Df de cointegrale seque

$$\mathrm{mod}(f\mathrm{x}^s n) = m^s \times v\mathrm{x}^s n \tag{1}$$

et si ϱ es maximo inter exponente de potestates de numeros primo que divide n, nos habe pro $s>0$ (P8·7):

$$v\mathrm{x}^s = \Pi[\mathrm{C}(\mathrm{mp}(x,n)+s-1,\ \mathrm{mp}(x,n)) \,|x, \mathrm{Np}]$$
$$\leqq [\mathrm{C}(\varrho+s-1, \varrho)] \upharpoonleft (\varkappa n) \tag{2}$$

et si nos pone

$$u \,\varepsilon\, \mathrm{N_1FN_0} \,.\, u_0 = 1 \,.\, u_s = [\mathrm{C}(\varrho+s-1, \varrho)] \upharpoonleft (\varkappa n)$$

tunc modulo de omni termine de serie

$$\Sigma(a_s \times f\mathrm{x}^s n \,|s, \mathrm{N}_0) \tag{3}$$

es minore de correspondente termine de serie

$$\Sigma(u_s \times \mathrm{mod}a_s \times m^s |s, \mathrm{N}_0)\,; \tag{4}$$

et si nos demonstra que serie (4) es convergente, seque, per P noto, convergentia absoluto de serie (3).

Ex Hp seque $m<r$ et nos pote sume numero c comprehenso inter m et r; tunc serie

$$\Sigma(\mathrm{mod}a_s \times c^s \,|s, \mathrm{N}_0) \tag{5}$$

es convergente et serie

$$\Sigma(u_s m^s / c^s \,|s, \mathrm{N}_0) \tag{6}$$

es et convergente, quare, si nos applica regula de CAUCHY, nos inveni

$$\lim[(u_{s+1} m^{s+1}/c^{s+1}) \,/\, (u_s m^s/c^s)] |s = m/c < 1\,.$$

Tunc serie (4), ubi termine generale es producto de termine generale de duo serie convergente (5) et (6), es convergente, ut nos debe-ba demonstra.

·1 Hp·0 . $z \varepsilon \mathrm{q}'$. mod$z<r$. $\psi = \Sigma(a_s z^s |s, \mathrm{N}_0) \,|z$.$\supset$.
$\psi(f\mathrm{x}n) = \Sigma(a_s \times f\mathrm{x}^s n |s\, \mathrm{N}_0)$ Df

·11 $\psi f\mathrm{x} = \psi(f\mathrm{x}n)|n$ Df

Si ψ es functione repræsentato per serie $\Sigma(a_s z^s |s, \mathrm{N}_0)$ in suo circulo de convergentia, nos indica cum $\psi(f\mathrm{x}n)$ summa de serie $\Sigma(a_s \times f\mathrm{x}^s n |s, \mathrm{N}_0)$ et dic que ce serie repræsenta « functione

ψ numerico integrale de f » pro omni numero n, que satisfac ad conditione:

$$d \varepsilon\, \mathrm{N_1} \frown n/\mathrm{N_1} \,.\!\supset_d .\ \mathrm{mod}(fd) < r.$$

·2 Hp·1 . $f \varepsilon (\mathrm{q}'\mathrm{FN_1})$comp .$\supset$. $\psi f\mathrm{x} = f \times \psi v\mathrm{x}$

Dm. Hp . $m \varepsilon \mathrm{N_0}$. P8·8 .$\supset$. $f\mathrm{x}^m = f \times v\mathrm{x}^m$.$\supset$. P.

Cum opportuno transformatione nos pote semper inveni summa de serie $\Sigma(a_s \times f\mathrm{x}^s n \,|s, \mathrm{N}_0)$. Sed nos obtine id ut consequentia de P sequente, magis generale, que exprime, ut nos ita dic, theorema de TAYLOR de Calculo numerico-integrale:

·3 Hp·1 . $g \varepsilon\, \mathrm{qFN_1}$: $d \varepsilon\, \mathrm{N_1} \frown n/\mathrm{N_1} \,.\!\supset_d .\ \mathrm{mod}(fd+gd) < r$:$\supset$.
$\psi((f+g)\mathrm{x}n) = \Sigma[(g\mathrm{x}^s/s! \;\mathrm{x}\; \mathrm{D}^s \psi f\mathrm{x})n \,|s, \mathrm{N}_0]$

In vero, ex Hp. et P·1 seque

$$\psi((f+g)\mathrm{x}n) = \Sigma[a_m \times (f+g)\mathrm{x}^m n | m, \mathrm{N}_0] =$$
$$= \Sigma[a_m \times \Sigma\{\mathrm{C}(m,s) \times (f\mathrm{x}^{m-s} \mathrm{x} g\mathrm{x}^s) n | s, \mathrm{N}_0 + m\} | m, \mathrm{N}_0]$$

Per proprietate noto, nos pote inverte summas in ce serie duplo et nos obtine

$$\psi((f+g)\mathrm{x}n) = \Sigma[(g\mathrm{x}^s \mathrm{x} \Sigma\{a_m \times \mathrm{C}(m,s) \times f\mathrm{x}^{m-s} | m, \mathrm{N}_0 + s\})n \,|s, \mathrm{N}_0] \tag{1}$$

Sed in circulo de convergentia de radio r, pro omni numero integro et positivo s, existe $\mathrm{D}^s\psi$ et es repræsentato per serie

$$\Sigma[a_m \times \mathrm{C}(m,s) \times s! \times z^{m-s} | m, \mathrm{N}_0 + s]$$

ergo ex P·1 seque

$$\mathrm{D}^s \psi(f\mathrm{x}n) = \Sigma[a_m \times \mathrm{C}(m,s) \times s! \times f\mathrm{x}^{m-s} n \,| m, \mathrm{N}_0 + s]$$

et æqualitate (1) fi

$$\psi((f+g)\mathrm{x}n) = \Sigma[(g\mathrm{x}^s/s! \;\mathrm{x}\; \mathrm{D}^s \psi f\mathrm{x})n \,|s, \mathrm{N}_0],$$

ut nos debeba demonstra.

Si nos in P præcedente muta f in $(f1) \times a$ et g in $f-(f1)\times a$, omni conditione es satisfacto et nos obtine

·4 Hp·1 .$\supset$. $\psi(f\mathrm{x}n) = \Sigma[\mathrm{D}^s\psi(f1) \times (f-(f1)\times a)\mathrm{x}^s n/s! \,|s, \mathrm{N}_0]$.

Membro 2 de ce æqualitate habe numero finito de termine, quare, quando gradu de cointegrale de $f-(f1)\times a$ supera valore de τn, valore de cointegrale es nullo. Ergo nos pote sume æqualitate ·4 ut Df de $\psi(f\mathrm{x}n)$, si ψ es functione analytico de variabile complexo z, holomorpho in regione que contine puncto $f1$.

Nos obtine ex P·4 plure P notabile:

·5 $s\varepsilon Q \,.\, f\varepsilon\, q'FN_1 \,.\!\supset.$
$f\mathrm{x}^s = \Sigma[(f-(f1)\times\alpha)\mathrm{x}^m \times \Pi(s-x|x,\, 0\cdots(m-1))/m! \times (f1)^{s-m}|m,\, N_0]$

·6 $s\varepsilon = -Q \,.\, f\varepsilon\, q'FN_1 \,.\, f1 -= 0 \,.\!\supset.$ Th·5

Si in ce P nos muta s in -1, nos inveni expressione de functione coniugato de f, dato iam per P10·3.

·7 $f\varepsilon\, q'FN_1 \,.\!\supset.$
$e^{f\mathrm{x}} = \Sigma(f\mathrm{x}^s/s!|s, N_0) = e^{f1} \times \Sigma[(f-(f1)\times\alpha)\mathrm{x}^s/s!|s,\, N_0]$

·8 $f\varepsilon\, q'FN_1 \,.\, f1 -= 0 \,.\!\supset.$
$\log f\mathrm{x} = \alpha\times\log(f1) + \Sigma[(-1)^{s-1}(f-(f1)\times\alpha)\mathrm{x}^s/(s\times(f1)^s)|s,\, N_1]$

Es utile demonstra que si f es functione numerico composito, $\log(f\mathrm{x}n)$ habe valore nullo pro omni numero n, que non es potestate de numero primo, et, si n es potestate de numero primo, valore de $\log(f\mathrm{x}n)$ es fn diviso per exponente de ce potestate. Nos pote exprime id per P sequente

·9 $f\varepsilon\, (q'FN_1)\mathrm{comp} \,.\, f1 -=0 \,.\, n\varepsilon\, N_1-(Np|N_1) \,.\, p\varepsilon Np \,.\, m\varepsilon N_1 \,.\!\supset.$
$\log(f\mathrm{x}n) = 0 \,.\, \log(f\mathrm{x}(p^m)) = f(p^m)/m$

Dem.

$Hp \,.\, P{\cdot}2 \,.\!\supset.\, \log f\mathrm{x}n = fn \times \log v\mathrm{x}n$ (1)
$Hp \,.\, P{\cdot}8 \,.\!\supset.\, \log(f\mathrm{x}1) = 0$ (2)
$Hp \,.\, a\varepsilon N_1+1 \,.\, (v|f)P{\cdot}8 \,.\!\supset.\, \log(v\mathrm{x}a) = \Sigma[(-1)^{s-1}\times(v-\alpha)\mathrm{x}^s\, a/s\,|s,\, 1\cdots\tau a]$
$= \Sigma[(-1)^{s-1}\times\Sigma\{(-1)^r/s\times C(s,r)v\mathrm{x}^{s-r}a\,|r,\, N_0\}\,|s,\, 1\cdots\tau a]$
$= \Sigma[(-1)^{s-1}/s\times C(\tau a,s)\times v\mathrm{x}^s\, a\,|s,\, 1\cdots\tau a]$ (3)
$Hp \,.\, p\varepsilon\, Np\cap a/N_1 \,.\, m = \mathrm{mp}(p,a) \,.\, P9{\cdot}0 \,.\!\supset.\, \alpha(a/p) =$
$\Sigma[(-1)^{s-1}\times C(\tau a,s)\times v\mathrm{x}^s\,(a/p)|s,\, 1\cdots\tau a]$ (4)
$(4) \,.\!\supset.\, \alpha(a/p) = \alpha(a/p^m)\times\alpha(p^{m-1}) =$
$= \Sigma[(-1)^{s-1}\times C(\tau a,s)\times v\mathrm{x}^s\,(a/p^m)\times v\mathrm{x}^s\,(p^{m-1})\,|s,\, 1\cdots\tau a]$ (5)
$Hp \,.\, P8{\cdot}7 \,.\!\supset.\, \int\alpha(h^{m-1}) = v(p^{m-1}) \,.\, \int v\mathrm{x}^s\,(p^{m-1}) = v\mathrm{x}^{s+1}(p^{m-1}) =$
$C(s+m-1,s) = (m/s)\times v\mathrm{x}^s\,(p^m)$ (6)
$Hp \,.\, (4) \,.\, (5) \,.\, (6) \,.\!\supset.\, \alpha(a/p^m) = \Sigma[(-1)^{s-1}\times C(\tau a,s)\times v\mathrm{x}^s\, a/s|s,\, 1\cdots\tau a]$ (7)
$(3) \,.\, (7) \,.\!\supset.\, \log(v\mathrm{x}a) = \alpha(a/p^m)/m$ (8)
$Hp \,.\, (1) \,.\, (2) \,.\, (8) \,.\, Df\alpha \,.\!\supset.\, P$

§ 11 Interpretatione de functiones analytico numerico-integrale in theoria de serie de Dirichlet.

Nos nunc demonstra relatione que functione analytico numerico-integrale de functione f habe cum serie de DIRICHLET $\Sigma(fn/n^x|n,\, N_1)$, ubi x es variabile imaginario. Sed antea nos puta utile, et pro futuro applicatione, refer propositiones fundamentale de theoria de ce serie.

* 17. $f\varepsilon\, q'FN_1 \,.\, x\varepsilon q' \,.\!\supset:$

·1 $a\varepsilon q' \,.\, \Sigma(fn/n^a|n,N_1)\varepsilon q' \,.\, \mathrm{real}x > \mathrm{real}a \,.\!\supset.\, \Sigma(fn/n^x|n,N_1)\, \varepsilon q'$

(JENSEN, Tidsskrift for Math. s.3, t.2, a.1884, p.70). Ex ce P deducere que regione de convergentia de serie de DIRICHLET es semiplano limitato per recta parallelo ad axi de iq. Abscissa de ce recta, nos appella « abscissa de convergentia de serie ». Abscissa de convergentia pote habe valore $+\infty$ (serie semper divergente) et $-\infty$ (serie semper convergente).

·2 $\mathrm{absc}\, f = \mathrm{l}_\prime\, \mathrm{real}\{q'\cap x\ni[\Sigma(fn/n^x\,|n,\, N_1)\,\varepsilon q']\}$ Df

Lege absc$f$ « abscissa de convergentia de serie . $\Sigma(fn/n^x|n,\, N_1)$, pro x ».

In regione de convergentia serie de DIRICHLET pote non es convergente in modo absoluto. Regione de convergentia absoluto es et semiplano limitato per recta parallelo ad axi de iq. Abscissa de ce recta es æquale ad absc mod$f$.

·3 $\mathrm{absc}\, f \leqq \mathrm{absc}\,\mathrm{mod}f$

·4 $\mathrm{absc}\, f \geqq 0 \,.\!\supset.$
$\mathrm{absc}\, f = \max \mathrm{Lm}[\log\mathrm{mod}\Sigma(f,1\cdots n)/\log n]\,|n$

·5 $\mathrm{absc}\,\mathrm{mod}f \geqq 0 \,.\!\supset.$
$\mathrm{absc}\,\mathrm{mod}f = \max\, \mathrm{Lm}[\log\Sigma(\mathrm{mod}f,\, 1\cdots n)\, /\, \log n]\,|n$

Exemplo:

·6 $\mathrm{absc}\, v = 1$.

·7 $\mathrm{absc}[(-1)^n|n] = 0$

Propositione fundamentale in theoria de serie de DIRICHLET:

·8 $c\varepsilon\mathrm{q} \,.\, c > \mathrm{absc}f \;.\supset:$
$h\varepsilon\mathrm{Q} \;.\supset_h.\; \exists\, \mathrm{N}_0 \frown p\ni\{x\varepsilon\mathrm{q}' \,.\, \mathrm{real}x \geqq c \,.\, q\varepsilon p+\mathrm{N}_0 \;.\supset_{x,q}.$
$\mathrm{mod}\Sigma(fn/n^x|n, p\cdots q) < h\}$

In phrasi: serie $\Sigma(fn/n^x|n,\mathrm{N}_1)$ in omni regione interiore, cum suo limite, ad semiplano de convergentia es convergente in modo uniforme. Ergo ex noto theorema de WEIERSTRASS (Monatsberichte der K. Preuss. Ak. der Wiss. zu Berlin, a.1880, p.723, Werke, t.2, a.1895, p.205) seque:

·9 $u\varepsilon\, \mathrm{Cls}'\mathrm{q}' \frown x\ni(\mathrm{real}x > \mathrm{absc}f) \,.\, u = \overline{\mathrm{F}}u \,.\, x\varepsilon u \,.$
$h = [\Sigma(fn/n^x|n, \mathrm{N}_1)\,|x, u] \;.\supset.\; h, \mathrm{D}h\, \varepsilon\, (\mathrm{q}'\mathrm{F}u)\mathrm{cont}$

Id es: in omni regione u, perfecto, interiore ad semiplano de convergentia, serie $\Sigma(fn/n^x\,|n, \mathrm{N}_1)$ defini functione h de variabile x holomorpho (= finito, continuo, uniforme vel monodromo) cum suo derivata.

Pro theoria de serie de DIRICHLET vide CAHEN, Ann. de l'École norm. sup., s.3, t.11, a.1894, p.75.

❋ 18. $f,g\varepsilon\, \mathrm{q}'\mathrm{FN}_1 \,.\, x\varepsilon\mathrm{q}' \;.\supset.$

·1 $\Sigma[\mathrm{mod}(fn/n^x)|n,\mathrm{N}_1],\ \Sigma(gn/n^x|n,\mathrm{N}_1)\ \varepsilon\mathrm{q}' \;.\supset.$
$\Sigma[(f\mathrm{x}g)n/n^x|n,\mathrm{N}_1] = \Sigma(fn/n^x|n,\mathrm{N}_1)\times\Sigma(gn/n^x|n,\mathrm{N}_1)$

In phrasi: si serie $\Sigma(fn/n^x|n,\mathrm{N}_1)$ es convergente in modo absoluto et serie $\Sigma(gn/n^x|n,\mathrm{N}_1)$, tunc es convergente serie DIRICHLET, que habe pro termine generale producto integrale de termine de duo serie dato, et suo summa es producto de summa de duo serie dato.

Ce P es casu particulare de theorema de STIELTJES, Nouv. Ann. de Math., s.3, t.6, a.1887, p.210-215. Vide et LANDAU, Rend. Circ. Mat. di Palermo, t.24, a.1907.

In particulare: si duo serie dato es convergente in modo absoluto, tunc serie producto supra definito es et convergente in modo absoluto:

·2 $\Sigma[\mathrm{mod}(fn/n^x)|n, \mathrm{N}_1],\ \Sigma[\mathrm{mod}(gn/n^x)|n, \mathrm{N}_1]\ \varepsilon\mathrm{Q} \;.\supset.$
$\Sigma[\mathrm{mod}\{(f\mathrm{x}g)n/n^x\}|n, \mathrm{N}_1]\ \varepsilon\mathrm{Q}$

·3 $\Sigma[fn/n^x|n, \mathrm{N}_1),\ \Sigma(gn/n^x|n, \mathrm{N}_0),\ \Sigma[(f\mathrm{x}g)n/n^x|n, \mathrm{N}_1]\varepsilon\mathrm{q}'$
$.\supset.\ \mathrm{Ths}{\cdot}1$

(Vide LANDAU, op. cit., et pro alio propositiones particulare)

·4 $\zeta = \Sigma(/n^x|n, \mathrm{N}_1)\,|x,\ \mathrm{q}'\frown x\ni(\mathrm{real}x > 1)$ Df

·2 $r\varepsilon\mathrm{N}_1 \,.\, \mathrm{absc}\,f = c \,.\, \mathrm{real}x > \max(\iota 1 \cup \iota c)$
$.\supset.\ \Sigma(\int^r fn/n^x) = (\zeta x)^r\times\Sigma(fn/n^x|n,\mathrm{N}_1)$

·51 $\mathrm{real}x > 1 \;.\supset.\; \Sigma(v\mathrm{x}^r n/n^x|n, \mathrm{N}_1) = (\zeta x)^r$

·52 $\mathrm{real}x > 1 \;.\supset.\; \Sigma(vn/n^x|n,\mathrm{N}_1) = (\zeta x)^2$

·53 $m\varepsilon\mathrm{Q} \,.\, \mathrm{real}x > m+1 \;.\supset.\; \Sigma(\sigma_m n/n^x|n,\mathrm{N}_1) = \zeta(x-m)\times(\zeta x)$

Si nunc ψ es functione holomorpho de variabile imaginario z in regione u que contine puncto $f1$, tunc ψz pote exprimere, pro omni puncto de circulo interiore ad regione u cum centro in $f1$, per serie

$$\psi z = \Sigma[(z-f1)^r\times\mathrm{D}^r\psi(f1)/r!|r, \mathrm{N}_0]$$

Si postea serie $\Sigma(fn/n^x|n,\mathrm{N}_1)$ habe convergentia absoluto per omni x interiore ad semiplano v, ubi (P17·9) repræsenta functione h holomorpho de variabile x, que sume valore $f1$, pro valore $+\infty$ de x, tunc nos pote dic que, quando x varia in regione w interiore ad semiplano v et continente punto $+\infty$, tali que valore hx es semper interiore ad circulo cum centro in $f1$ et interiore ad regione u, ψh es functione holomorpho de x, et pro puncto de regione w es

$$\psi(hx) = \Sigma[(hx-f1)^r\times\mathrm{D}^r\,\psi(f1)/r!\ |r, \mathrm{N}_0] \qquad (1)$$

Sed in omni puncto x de semiplano v, ideo de regione w, es

$$hx - f1 = \Sigma[(fn/n^x|n, \mathrm{N}_1) - \Sigma[(f1)\times(an)/n^x|n, \mathrm{N}_1] =$$
$$= \Sigma[((fn)-(f1)\times(an))^r/n^x|n, \mathrm{N}_1],$$

ergo per omni x interiore ad semiplano v, ideo ad regione w, es

$$(hx-f1)^r = \Sigma[(f-(f1)\times a)\mathrm{x}^r n/n^x|n,\mathrm{N}_1],$$

et æqualitate (1) fi:

$$\psi(hx) = \Sigma[\mathrm{D}^r\,\psi(f1)/r!\times\Sigma\{(f-(f1)\times a)\mathrm{x}^r n/n^x\ |n, \mathrm{N}_1\}\ |r, \mathrm{N}_0]$$

Per convergentia absoluto de serie (1) nos pote commuta ordine de summa in ce serie duplo et nos obtine:

$$\psi(hx) = \Sigma[(/n^x)\times\dot{\Sigma}\{D^r\,\psi(f1)\times(f-(f1)\times a)\mathrm{x}^r n/r!\,|r, N_0\}|n, N_1]$$

et per P16·4:

$$\psi(hx) = \Sigma[\psi(f\mathrm{x}n)/n^x|n, N_1] \qquad (2)$$

Ergo: si ψ es functione holomorpho in regione u que contine puncto $f1$ et si serie $\Sigma(fn/n^x|n, N_1)$ per omni x interiore ad semiplano v repræsenta functione h holomorpho de variabile x, tunc existe regione w de semiplano v, que contine puncto $+\infty$, ubi ψh es functione holomorpho de x et serie $\Sigma[\psi(f\mathrm{x}n)/n^x|n, N_1]$ es convergente in modo absoluto et uniforme et suo summa es $\psi(hx)$.

In symbolo:

·6 $u\varepsilon\,\mathrm{Cls'q'} \,.\, u = \bar{\nabla} u \,.\, f1\,\varepsilon u \,.\, \psi, D\psi\,\varepsilon\,(\mathrm{q'F}u)\mathrm{cont}\,.$

$c\varepsilon\mathrm{q} : x\varepsilon\mathrm{q'} \,.\, \mathrm{real}x > c \,.\supset_x.\, \Sigma[\mathrm{mod}(fn/n^x)|n, N_1]\,\varepsilon\mathrm{Q} :\supset.$

$\exists\, \mathrm{Cls'q'}\frown w\ni\{w = \bar{\nabla} w : x\varepsilon w \,.\supset_x.\, \psi(hx) = \Sigma[\psi(f\mathrm{x}n)/n^x\,|n, N_1]\}$

Sed serie (2) es convergente in particulare semiplano et repræsenta functione holomorpho, ergo si functione ψh es holomorpho in omni regione interiore ad ce semiplano, summa de serie (2), pro omni x interiore ad ce semiplano, es $\psi(hx)$.

Seque ex P præcedente que si functione ψ es regulare in omni regione de plano et serie $\Sigma(fn/n^x|n,N_1)$ es convergente in modo absoluto in semiplano v, ubi repræsenta functione h holomorpho, tunc ψh es functione holomorpho in omni regione interiore ad semiplano v, que non contine puncto pro que h sume valore que es polo de ψ.

Sed si serie (2) es convergente in omni regione interiore ad semiplano v, tunc pro omni puncto interiore ad semiplano v es $\psi(hx) = \Sigma(fn/n^x|n, N_1)$.

Per exemplo, si nos sume $\psi = \log z|z$, et si functione f es composito et serie $\Sigma(fn/n^x|n, N_1)$ es convergente in modo absoluto in semiplano v, existe regione w de v, tali que pro omni x interiore ad w es

$$\log\Sigma(fn/n^x|n, N_1) = \Sigma(\log(f\mathrm{x}n)/n^x|n, N_1) \qquad (1)$$

et per P16·9 es

$$\log\Sigma(fn/n^x|n, N_1) = \Sigma[\Sigma\{(fp)^r/(rp^{rx})|r, N_0\}|p, \mathrm{Np}] =$$
$$= -\Sigma[\log(1-fp/p^x)|p, \mathrm{Np}] = \log\Pi[/(1-fp/p^x)|p, \mathrm{Np}] \qquad (2)$$

Sed ex convergentia absoluto de serie $\Sigma(fn/n^x|n, N_1)$ per omni x interiore ad semiplano v seque convergentia absoluto de serie $\Sigma(fp/p^x|p, \mathrm{Np})$ per omni x interiore ad semiplano v, et pro noto theorema de WEIERSTRASS (Form. 1905, V, §1, P60·6) $\Pi[1-fp/p^x|p, \mathrm{Np}]$ es convergente in modo absoluto et uniforme et non habe valore nullo pro omni x interiore ad semiplano v.

Ergo æqualitate (1) et (2) subsiste in omni puncto interiore ad semiplano v.

Ex æqualitate (2) seque P noto:

·7 $f\varepsilon\,(\mathrm{q'F}N_1)\mathrm{comp} \,.\, c\varepsilon\mathrm{q} \,.\, \mathrm{real}x > c \,.\, \Sigma[\mathrm{mod}(fn/n^x)n, N_1]\,\varepsilon\mathrm{Q}$
$\quad .\supset.\, \Sigma(fn/n^x|n, N_1) = \Pi[/(1-fp/p^x)|p, \mathrm{Np}]$

In particulare ($(v|f)$P·7) seque

·61 $\mathrm{real}x > 1 \,.\supset.\, \zeta x = \Pi[/(1-/p^x)|p, \mathrm{Np}]$

Ce P, cum Hp: $x\varepsilon 1+\mathrm{Q}_0$, es dato ab EULERO, a.1744, Petr. C., t.9, p.122; a.1748, p.225; vide Form. 1905; V. §1, P37·6.

·72 $\mathrm{real}x > 1 \,.\supset.\, \Sigma[(-1)^{\frown}(\tau n)/n^x\,|n, N_1] =$
$\qquad \Pi[/(1+/p^x)|p, \mathrm{Np}] = \zeta(2x)/(\zeta x)$

·8 $\mathrm{real}x > 1 \,.\supset.\, /(\zeta x) = \Sigma(\mu n/n^x|n, N_1)$

·81 $r\varepsilon N_1 \,.\, \mathrm{absc}\,f = c \,.\, \mathrm{real}x > \max(\iota 1 \cup \iota c)$
$\quad .\supset.\, \Sigma(\partial^r fn/n^x|n,N_1) = \Sigma(fn/n^x|n,N_1)/(\zeta x)^r$

·9 $\mathrm{real}x > 2 \,.\supset.\, \Sigma(\Phi n/n^x|n,N_1) = \zeta(x-1)/(\zeta x)$

Dm. $((x|x)|f, 1|r)$P·81 . P12·43 $.\supset.$ P

Nos in ce § tange vix theoria de serie de DIRICHLET, sed nos reverte ad illo in Applicationes.

Panormo, 19 augusto 1907.

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